ANALYTICAL METHOD OF MECHANICS IN THE THEORY OF A PERFECTLY ELASTIC IMPACT OF MATERIAL SYSTEMS

(ANALITICHESKII METOD MEKHANIKI V TEORII ABSOLIUTNO UPRUGOGO UDARA MATERIAL'NYKH SISTEM)

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The investigation of the motion of material systems on the basis of the fundamental equation of mechanics (the analytical method in mechanics) has certain definite advantages because it permits one to make such an investigation without the consideration of the reactions of constraints within the system, which in the majority of cases considerably simplifies the problem. This applies especially to the investigation of impact, in particular of impact which occurs in a system with one-sided constraints. The application of the fundamental equation of mechanics to the latter case leads, however, to a difficulty which is caused by the ambiguity in the relations between the equations of impact thus obtained. In [1] it is shown how this difficulty can be overcome in the simplest case when the system is subjected to a constraint which is expressed by a single analytic inequality which does not depend explicitly on time. The aim of the present study is to extend this method to the general case.

1. A system of n material points with masses m_i, joined by smooth constraints

$$\sum p_{\alpha i} v_i + p_\alpha = 0 \tag{1.1}$$

is moving in space relative to a fixed rectangular coordinate system ($m_1 = m_2 = m_3$ is the mass of the first point; $m_4 = m_5 = m_6$ is the mass of the second point, and so on; v_1, v_2 and v_3 are the velocity components of the first point, v_4, v_5 and v_6 those of the second point, etc.).

At a certain instant during the motion there is imposed upon the system some smooth one-sided constraint

$$\sum r_{\beta i} v_i + r_\beta \geqslant 0 \tag{1.2}$$

Then there occurs in the system an impact and in consequence of it a discontinuous change of the velocities of its points.

Note. Equations (1.1) and (1.2) are taken as velocity-dependent. This, obviously, does not mean that all relations depend on the velocities. However, for the sake of brevity of presentation it is convenient to reduce all the equations to the indicated form. The notation used for the variables is adopted for the same reason.

Treating an impact as a process which lasts a very short time but is still of a definite duration, we assume that during the impact the relations (1.2) prevail in the system and that the fundamental equation of mechanics is valid. Then it is not difficult to show that for the case of an ideal impact whose duration is zero (classical mechanics deals only with this case) the fundamental equation of mechanics takes on the form

$$\sum m_i (v_i - v_{i0}) \delta x_i = 0 \tag{1.3}$$

where the v_{i0} represent the velocities of the system before the impact. The "possible displacements" δx_i of the system satisfy the relations

$$\sum p_{\alpha i} \delta x_i = 0, \quad \sum r_{\beta i} \delta x_i = 0 \tag{1.4}$$

From this one can derive a number of equations of impact. However, the addition to them of Equation (1.1) does not close the system of equations. The number of equations lacking is equal to the number of equations which are imposed upon the system by the one-sided constraints. The equations of the one-sided constraints themselves cannot be used to close the system of impact equations, for they do not give sufficient conditions for the determination of the state of the system after the impact. Hence, the above-made assumptions relative to the impact are insufficient and one needs to find supplementary relations.

Note. We have met a similar situation already in [2]. There, one equation was lacking for the closure of the system of equations of impact. The condition of the conservation of the kinetic energy of the system during impact was used there to obtain the supplementary equation. This was quite natural because neither the connections within the system nor the connections imposed on the system depended on time in [2]. In the present study such a solution of the problem does not apply, for here we admit a dependence on time for the constraints. Furthermore, we deal here with the case when the imposed constraints are expressed by means of certain inequalities and, hence, several additional conditions are needed to close the system of equations.

Thus, under the hypothesis of the present work one cannot make a direct application of the analytical method of investigating an impact for it is not clear in the beginning how one can close the system of equations of the given setup. Hence, one has to start with a different principle. For this purpose we use, as was done in [3], the following hypothesis.

A perfectly elastic impact which occurs in a system subjected to one-sided constraints consists of a succession of two phases: the first, passive, phase when the impact proceeds inelastically and when there occurs an accumulation of reactions, and the second phase, when due to the reaction impulses accumulated during the first phase, there occurs an explosive release of the system from the one-sided constraints.

The possibility of such an interpretation of a perfectly elastic impact was established in [1] for the particular case of a system whose constraints are independent of time and which is subjected to an external constraint expressed by a single inequality which also is independent of time. The physical interpretation of that discussion indicates that the same type of conditions may prevail in the general case of an impact. In fact, it is for this reason that the above-formulated hypothesis is taken here as the general definition of a perfectly elastic impact. The same type of definition is suggested by the independence of the condition of impact on any assumptions with respect to the quantity and quality (in the sense of dependence on time) of the one-sided constraints imposed on the system. This makes it possible even in the most general case to evaluate the magnitudes of the changes which occur in consequence of the impact in the velocities of the system's points, and then to indicate the thus far unknown method of closing the set of impact equations on the basis of the general equation of mechanics.

2. In accordance with the definition of a perfectly elastic impact adopted here, let us consider two of its phases in succession. In this we shall follow the procedure used in [1].

During the first phase, the impact is totally inelastic, namely, all the constraints (1.2) imposed on the system remain in effect until the end of this phase. This means that at the end of the first phase, the velocities u_i of the system satisfy the relation

$$\sum r_{\beta i} u_i + r_\beta = 0 \tag{2.1}$$

Further, it is obvious that the following relations must hold:

$$\sum p_{\alpha i} u_i + p_\alpha = 0 \tag{2.2}$$

because the equations of the constraints of the system cannot be violated during the impact.

At the end of the first phase of the impact the state of the system is described by the equation

$$\sum m_i (u_i - v_{i0}) \delta x_i = 0 \tag{2.3}$$

in which the v_{i0} denote the velocities before the impact, and where the "possible displacements" δx_i satisfy the conditions

$$\sum p_{\alpha i} \delta x_i = 0, \qquad \sum r_{\beta i} \delta x_i = 0$$

Note. It may happen that the number of one-sided constraints imposed on the system is so large that the total number of Equations (1.1) and (1.2) is equal to or exceeds $3n$. For example, a material point moving freely in space may be connected by means of flexible inextensible threads to an arbitrary number of points of this space in such a way that at certain instances during the motion three or more threads may become tight. In this case the velocities of the system at the end of the first phase are determined simply by Equations (2.1) and (2.2).

From Equation (2.3) and from the relations for the "possible displacements" it follows directly that

$$m_i (u_i - v_{i0}) + \sum \lambda_\alpha p_{\alpha i} + \sum \mu_\beta r_{\beta i} = 0 \tag{2.4}$$

where λ_α and μ_β are undetermined factors. Let us find them. For this purpose we substitute into Equations (2.2) the expressions of u_i from (2.4), and obtain

$$\sum a_{\alpha\gamma} \lambda_\alpha + \sum b_{\beta\gamma} \mu_\beta = 0$$

Here

$$a_{\alpha\gamma} = \sum \frac{1}{m_i} p_{\alpha i} p_{\gamma i}, \qquad b_{\beta\gamma} = \sum \frac{1}{m_i} r_{\beta i} p_{\gamma i} \tag{2.5}$$

Taking into account the fact that the determinant $A = |a_{\alpha\gamma}|$ of the last system is different from zero (this can be established in a manner analogous to the proof given in [2]), we deduce that

$$\lambda_\alpha = -\sum \frac{A_{\gamma\alpha}}{A} b_{\beta\gamma} \mu_\beta$$

Here $A_{\gamma\alpha}$ is the algebraic cofactor of the element $a_{\alpha\gamma}$ in the determinant A. The last expression makes it possible to exclude from (2.4) the undetermined factors λ_α. Then the system (2.4) is reduced to the form

$$u_i - v_{i0} = \sum R_{i\beta} \mu_\beta, \qquad R_{i\beta} = \frac{1}{m_i} \left(\sum \frac{A_{\gamma\alpha}}{A} b_{\beta\gamma} p_{\alpha i} - r_{\beta i} \right) \tag{2.6}$$

We note that the quantities $R_{i\beta}$ satisfy the identity

$$\sum p_{\alpha i}R_{i\beta}=0, \qquad \sum r_{\delta i}R_{i\beta}=-\sum m_i R_{i\delta}R_{i\beta} \tag{2.7}$$

In order to establish this it is necessary to repeat for the quantities $R_{i\beta}$ the argument used in [1].

The equations (2.6) yield explicit expressions for the u_i in terms of the factors μ_β. In order to find the latter we substitute into Equations (2.1) the values of u_i from (2.6). We then obtain the system

$$\sum c_{\delta\beta}\mu_\beta+\sum r_{\delta i}v_{i0}+r_\delta=0 \qquad \left(c_{\delta\beta}=\sum r_{\delta i}R_{i\beta}\right) \tag{2.8}$$

We shall show that its determinant $C=|c_{\delta\beta}|$ is different from zero.

Indeed, let us assume that it is zero. Then one can find values k_δ, not all zero, such that

$$\sum c_{\delta\beta}k_\delta=\sum r_{\delta i}R_{i\beta}k_\delta=0$$

From this and from the identity (2.7) we obtain

$$\sum m_i R_{i\beta}\psi_i=0 \qquad \left(\psi_i=\sum R_{i\delta}k_\delta\right) \tag{2.9}$$

On the other hand, we have

$$\sum m_i\psi_i^2=\sum k_\delta R_{i\delta}m_i\psi_i$$

Hence, from the identity (2.9) it follows that $\Sigma m_i\psi_i^2=0$, or $\psi_i=0$.

Thus

$$\psi_i=\sum R_{i\delta}k_\delta=0$$

These equations can be rewritten in the form

$$\sum p_{\alpha i}l_\alpha+\sum r_{\delta i}k_\delta=0 \qquad \left(l_\alpha=-\sum\frac{A_{\gamma\alpha}}{A}\,b_{\delta\gamma}k_\delta\right)$$

Not all of the l_α and k_δ are zero (for the k_δ this is true by hypothesis). Therefore, these must exist a linear dependence among Equations (2.1) and (2.2), which is impossible. This means that our assumption relative to the determinant C is false, and the determinant is not zero as was to be proved. Taking into account this established fact, we derive from (2.8) the result

$$\mu_\beta=-\sum\frac{C_{\beta\delta}}{C}\left(\sum r_{\delta i}v_{i0}+r_\delta\right) \tag{2.10}$$

where $C_{\beta\delta}$ denotes the algebraic cofactor of the element $c_{\delta\beta}$ of the determinant C.

Thus, we find that at the end of the first (passive) phase of the impact the state of the system is described by Equations (2.6) in which μ_β has the form (2.10). The second phase of the impact consists, according to the definition, of the state during which the system is subjected to impulses of the reactions which were accumulated during the time of the first phase and were caused by the constraints (1.1) and (1.2). Here it is assumed that at the beginning of the second phase, the homogeneity of the constraints (1.2) has been re-established. The velocities v_i at the end of the second phase of the impact produce a minimum for the quantity

$$\sum \frac{m_i}{2}\left(v_i - u_i - \sum R_{i\beta}\mu_\beta\right)^2 \tag{2.11}$$

under the conditions that

$$\sum p_{\alpha i} v_i + p_\alpha = 0, \qquad \sum r_{\beta i} v_i + r_\beta \geqslant 0 \tag{2.12}$$

In accordance with the method presented in [1], we drop the inequalities in (2.12). Then the unknown v_i can be found from the equations

$$m_i\left(v_i - u_i - \sum R_{i\beta}\mu_\beta\right) + \sum \lambda_\gamma p_{\gamma i} = 0 \tag{2.13}$$

where λ_γ are undetermined coefficients. For their determination let us eliminate from Equations (2.12) the quantities v_i and obtain

$$\sum a_{\alpha\gamma}\lambda_\gamma - \sum p_{\alpha i} u_i - \sum p_{\alpha i} R_{i\beta}\mu_\beta = p_\alpha$$

where the $a_{\alpha\gamma}$ have the meaning of (2.5). In view of (2.2) and (2.7), the free terms of this system are zero; the determinant A of this system is not zero. From this it follows that all the λ_γ are zero. Thus, from (2.13) we have

$$v_i - u_i = \sum R_{i\beta}\mu_\beta \tag{2.14}$$

These equations describe the actual state of the system after the impact if the values of v_i determined by these equations satisfy the inequalities (2.12). But this they do. In fact, the values v_i from (2.14) satisfy the relations

$$\sum r_{\beta i} v_i + r_\beta = -\sum r_{\beta i} v_{i0} - r_\beta \tag{2.15}$$

One can establish this by substituting into (2.15) the expressions for the v_i from (2.14) and utilizing then the inequalities (2.1) and (2.8). On the other hand, before the impact we have the inequalities

$$\sum r_{\beta i}v_{i0} + r_\beta \leqslant 0$$

Using them in connection with (2.15) we can establish at once the correctness of the inequalities (2.12), which was to be proved.

Thus, the actual state of the system after the impact is described by Equations (2.14). Eliminating in them and Equations (2.16) the intermediate velocities u_i, we obtain the final formulas

$$v_i - v_{i0} = 2\sum R_{i\beta}\mu_\beta \tag{2.16}$$

which determine the state of the system after the impact in terms of its state before the impact.

3. It was mentioned above that the state of the system after the impact satisfies the relations (2.15).

We shall show that in contrast to other states which are admissible by the constraints of the system and which satisfy the relations (2.15), the state of the system after impact is such that for it, for all δx_i subjected to the relations (1.4), the condition (1.3) is fulfilled. For the proof one has, obviously, to show that the formulated conditions determine a unique state of the system and that the latter is precisely the state of the system after the impact. It must, however, be noted that there is no need to present here all necessary arguments, for they are practically the same as those given in the preceding section when we analysed the first phase of the system. It is only necessary to replace Equations (2.1) by the relations (2.15). This means that the difference will be only in the manner of determining the coefficients in Equations (2.6).

Thus the state of the system determined by the above-formulated conditions is described by the equations

$$v_i - v_{i0} = \sum R_{i\beta}\nu_\beta \tag{3.1}$$

where the undetermined factors ν_β must be found with the aid of relations (2.15). Eliminating the v_i in (2.15) with the aid of (3.1), we obtain

$$\sum r_{\delta i}v_{i0} + \sum c_{\delta\beta}\nu_\beta + r_\beta = -\sum r_{\delta i}v_{i0} - r_\delta$$

where the $C_{\delta\beta}$ have the same meaning as in Equations (2.8), and, furthermore

$$\sum c_{\delta\beta}\nu_\beta + 2\left(\sum r_{\beta i}v_{i0} + r_\delta\right) = 0$$

This shows that this system coincides, except for a coefficient of the free terms, with the system (2.8). Hence,

$$\nu_\beta = 2\mu_\beta \tag{3.2}$$

where the μ_β are taken from (2.10). Comparing (3.1) with (2.16), and taking into account (3.2), we see that the velocities after impact given by (3.1) coincide with the velocities determined by Equations (2.16). This establishes the following theorem.

Theorem. Among the various states of a system which are admissible by the constraints within the system and which satisfy the relations (2.15), the actual state after impact is the one for which the equation

$$\sum m_i (v_i - v_{i0})\, \delta x_i = 0$$

is satisfied for all displacements δx_i subjected to the relations

$$\sum p_{\alpha i} \delta x_i = 0, \qquad \sum r_{\beta i} \delta x_i = 0$$

Corollary. Among the various states of a system which are admissible by the constraints within the system and which satisfy relations (2.15), the actual state after impact is the one for which the function

$$\sum \frac{m_i}{2} (v_i - v_{i0})^2$$

attains a minimum value.

The theorem of this section establishes the possibility of the application of the fundamental equation of mechanics for the description of an impact in a material system subjected to one-sided constraints. The conditions (2.15) which define a perfectly elastic impact can be interpreted as consequences of elastic properties of the imposed constraints. Then the application of the fundamental equation becomes especially clear. At the moment of impact under the conditions of one-sided constraints within the system, there are imposed on the systems velocities supplementary restrictions. Thus, the impact occurs in such a way that the fundamental equation of mechanics (for impact) is satisfied for all "possible displacements" of the system. The relations for the "possible displacements" is obtained by the usual rules from Equations (1.1) and conditions (2.15).

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